

Cliques and Chromatic Number in Inhomogenous Random Graphs

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Abstract

In this paper, we study cliques and chromatic number of inhomogenous random graphs where the individual edge probabilities could be arbitrarily low. We use a recursive method to obtain estimates on the maximum clique size under a mild positive average edge density assumption. As a Corollary, we also obtain uniform bounds on the maximum clique size and chromatic number for homogenous random graphs for all ranges of the edge probability p_n satisfying $\frac{1}{n^{\alpha_1}} \leq p_n \leq 1 - \frac{1}{n^{\alpha_2}}$ for some positive constants α_1 and α_2 .

Key words: Random graphs, inhomogenous edge probabilities, cliques, chromatic number.

1 Introduction

Let K_n be the labelled complete graph on n vertices with vertex set $\{1, 2, \dots, n\}$ and edge set $\{e_1, e_2, \dots, e_m\}$, where $m = \binom{n}{2}$. Let $G_n = G(n, p_n)$ be the random graph obtain when every edge is independently open with probability $p_n \in (0, 1)$ and closed otherwise. Let $X(i, j)$ be a Bernoulli

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random variable defined on the probability space $(\{0, 1\}, \mathbb{B}(\{0, 1\}), \mathbb{P}_{i,j})$ with

$$\mathbb{P}_{i,j}(X(i, j) = 1) = p(i, j) = 1 - \mathbb{P}_{i,j}(X(i, j) = 0).$$

Here $\mathbb{B}(\{0, 1\})$ is the set of all subsets of $\{0, 1\}$. We say that edge $e(i, j)$ is *open* if $X(i, j) = 1$ and closed otherwise. The random variables $\{X(i, j)\}$ are independent and the resulting random graph G is an inhomogenous Erdős-Rényi (ER) random graph, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Here $\Omega = \{0, 1\}^{\binom{n}{2}}$, the sigma algebra \mathcal{F} is the set of subsets of Ω and $\mathbb{P} = \prod_{i,j} \mathbb{P}_{i,j}$.

Clique Number

Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Suppose that $\#V = n$ so that G is a graph on n vertices. We say that G is a *complete graph* if $\#E = \binom{n}{2}$. For a subset $U \subset V$ containing $\#U = r \leq n$ vertices, we define $G|_U = (U, E_U)$ to be the *induced subgraph* of G with vertex set U , defined as follows. For any two vertices $a, b \in U$, the edge e_{ab} with endvertices a and b belongs to E_U if and only if $e_{ab} \in E(G)$. We say that $G|_U$ is a *clique* if $G|_U$ is a complete graph. We denote $\omega(G)$ to be the size of the largest clique in G . Throughout, the size of a graph will always refer to the number of vertices in the graph.

Let $\mathbf{p} = \{p(i, j)\}_{i,j}$ be a vector of probabilities as defined in the previous subsection and let $G(n, \mathbf{p})$ be the resulting random graph. The following obtains an upper bound for the clique number $\omega(G(n, \mathbf{p}))$.

Proposition 1. *For any sequence $U_n > 0$ define*

$$\log \left(\frac{1}{t_n} \right) := \inf_{S: \#S=U_n} \left(\binom{\#S}{2} \right)^{-1} \sum_{i,j \in S} \log \left(\frac{1}{p(i, j)} \right). \quad (1.1)$$

We have

$$\mathbb{P}_{\mathbf{p}}(\omega(G(n, \mathbf{p})) \leq U_n) \geq 1 - \exp(-f_n U_n) \quad (1.2)$$

where

$$f_n := \frac{(U_n - 1)}{2} \log \left(\frac{1}{t_n} \right) - \log n \quad (1.3)$$

for all $n \geq 2$.

To obtain a lower bound on the clique number, we have a few definitions first. As before, let \mathbf{p} be the vector formed by the probabilities $\{p(i, j)\}_{i, j}$. For $p_n > 0$ and constant $0 \leq a < 1$, let $\mathcal{N}(a, n, p_n)$ be the set of all vectors \mathbf{p} satisfying the following condition: There is a constant $N = N(a) \geq 1$ such that for all $n \geq N$, we have

$$\inf_{1 \leq i \leq n} \inf_S \frac{1}{\#S} \sum_{j \in S} p(i, j) \geq p_n. \quad (1.4)$$

Here $N = N(a) \geq 1$ is a constant not depending on n . For a fixed $1 \leq i \leq n$, the infimum above is taken over all sets S such that $\#S \geq n^a$ and $i \notin S$. The condition implies that the average edge density taken over sets of cardinality at least n^a , is at least p_n . All constants mentioned throughout are independent of n .

Let p_n be as in (1.4) and define

$$\alpha_1 = \limsup_n \frac{\log \left(\frac{1}{p_n} \right)}{\log n} \quad (1.5)$$

and

$$\alpha_2 = \limsup_n \frac{\log \left(\frac{1}{1-p_n} \right)}{\log n}. \quad (1.6)$$

We consider three cases separately depending on whether $\alpha_1 > 0$ or $\alpha_2 > 0$ or both $\alpha_1 = \alpha_2 = 0$.

Theorem 1. *Suppose $\mathbf{p} \in \mathcal{N}(a, n, p_n)$ for some constant $0 \leq a < 1$.*

(i) *Suppose $0 < \alpha_1 < 2$ and let $\eta, \gamma > 0$ be such that*

$$\max \left(\frac{\alpha_1}{2}, a \right) + \gamma < \eta < 1. \quad (1.7)$$

We have that $\alpha_2 = 0$ and there is a positive integer $N_1 = N_1(\eta, \gamma, \alpha_1, a) \geq 1$ so that

$$\mathbb{P}_{\mathbf{p}} \left(\omega(G(n, \mathbf{p})) \geq (1 - \eta) \frac{\log n}{\log \left(\frac{1}{p_n} \right)} \right) \geq 1 - 3 \exp(-n^{2\eta - 2\gamma - \alpha_1}) \quad (1.8)$$

for all $n \geq N_1$.

(ii) *Suppose $\alpha_1 = \alpha_2 = 0$. Let $\eta, \gamma > 0$ be such that*

$$a < \gamma < \eta < 1. \quad (1.9)$$

There is a positive integer $N_2 = N_2(\eta, \gamma) \geq 1$ so that

$$\mathbb{P}_{\mathbf{p}} \left(\omega(G(n, \mathbf{p})) \geq (1 - \eta) \frac{\log n}{\log \left(\frac{1}{p_n} \right)} \right) \geq 1 - 3 \exp(-n^{2\eta-2\gamma}) \quad (1.10)$$

for all $n \geq N_2$.

(iii) Suppose $0 < \alpha_2 < 1$ and let $\eta, \gamma > 0$ be such that

$$\max \left(\gamma - \frac{\alpha_2}{2}, a \right) < \eta < 1 - \alpha_2. \quad (1.11)$$

We have that $\alpha_1 = 0$ and there is a positive integer $N_3 = N_3(\eta, \gamma) \geq 1$ so that

$$\mathbb{P}_{\mathbf{p}} \left(\omega(G(n, \mathbf{p})) \geq (1 - \alpha_2 - \eta) \frac{\log n}{\log \left(\frac{1}{p_n} \right)} \right) \geq 1 - 3 \exp(-n^{2\eta-2\gamma+\alpha_2}) \quad (1.12)$$

for all $n \geq N_3$.

The usual method for studying the lower bound for clique numbers of homogenous random graphs uses a combination of second moment method and martingale inequalities (see for e.g., Alon and Spencer (2003), Bollobas (2001)). For inhomogenous graphs where the edge probabilities could be arbitrarily low, the above method is not directly applicable. We use a recursive method to obtain lower bounds on the clique numbers (see Lemma 5, Section 2).

As a consequence of our main Theorem above, we also obtain results for homogenous random graphs where the edge probabilities $p(i, j) = p_n$ for $1 \leq i \neq j \leq n$.

Proposition 2. *If $\alpha_1 > 2$, then fix $\epsilon > 0$ small so that $\alpha_1 - \epsilon > 2$. We then have*

$$\mathbb{P}(\omega(G(n, p_n)) = 1) \geq 1 - \frac{1}{n^{\alpha_1 - \epsilon - 2}} \quad (1.13)$$

for all n large. Similarly, if $\alpha_2 > 2$, then fix $\epsilon > 0$ small so that $\alpha_2 - \epsilon > 2$. We then have

$$\mathbb{P}(\omega(G(n, p_n)) = n) \geq 1 - \frac{1}{n^{\alpha_2 - \epsilon - 2}} \quad (1.14)$$

for all n large. If $1 < \alpha_2 < 2$, then fix $\epsilon > 0$ small so that $0 < 2 - \alpha_2 - 2\epsilon < 2 - \alpha_2 + 2\epsilon < 1$. We then have

$$\mathbb{P}(\omega(G(n, p_n)) \geq n - n^{2-\alpha_2+2\epsilon}) \geq 1 - \exp(-n^{2-\alpha_2-2\epsilon}) \quad (1.15)$$

for all n large.

Let $f_n \rightarrow \infty$ be any sequence and let

$$U_n = \frac{2 \log n + 2f_n}{\log\left(\frac{1}{p_n}\right)} + 1 \geq 1. \quad (1.16)$$

We have

$$\mathbb{P}(\omega(G(n, p_n)) \leq U_n) \geq 1 - \exp(-f_n U_n) \quad (1.17)$$

for all $n \geq 2$.

We have the following result regarding the clique number for the cases where $\alpha_1 < 1$ and $\alpha_2 < 1$.

Corollary 2. (i) Suppose $p_n = \frac{1}{n^{\theta_1}}$ for some $0 < \theta_1 < 1$. Fix $\eta, \gamma > 0$ such that $\frac{\theta_1}{2} + \gamma < \eta < 1$ and $\xi > 0$. There is a positive integer $N_1 = N_1(\eta, \gamma, \xi) \geq 1$ so that

$$\mathbb{P}_p\left(\frac{1-\eta}{\theta_1} \leq \omega(G(n, p_n)) \leq \frac{(2+\xi)}{\theta_1} + 1\right) \geq 1 - 3 \exp(-n^{2\eta-2\gamma-\theta_1}) - n^{-\frac{\xi(2+\xi)}{\theta_1}} \quad (1.18)$$

for all $n \geq N_1$.

(ii) Suppose $p_n = p \in (0, 1)$ for all n . Fix $0 < \gamma < \eta < 1$ and $\xi > 0$. There is a positive integer $N_2 = N_2(\eta, \gamma, \xi) \geq 1$ so that

$$\begin{aligned} & \mathbb{P}\left(\frac{(1-\eta) \log n}{\log\left(\frac{1}{p}\right)} \leq \omega(G(n, p_n)) \leq \frac{(2+\xi) \log n}{\log\left(\frac{1}{p}\right)}\right) \\ & \geq 1 - 3 \exp(-n^{2\eta-2\gamma}) - \exp\left(\frac{-\xi(1+\xi)}{\log\left(\frac{1}{p}\right)} (\log n)^2\right) \end{aligned} \quad (1.19)$$

for all $N \geq N_2$.

(iii) Suppose $p_n = 1 - \frac{1}{n^{\theta_2}}$ for some $0 < \theta_2 < 1$. Fix $\eta, \gamma > 0$ so that $\gamma - \frac{\theta_2}{2} <$

$\eta < 1 - \theta_2$ and fix $\xi > 0$. There is a positive integer $N_3 = N_3(\eta, \gamma, \xi) \geq 1$ so that

$$\begin{aligned} \mathbb{P} \left((1 - \theta_2 - \eta)n^{\theta_2} \log n \leq \omega(G(n, p_n)) \leq (2 + \xi)n^{\theta_2} \log n \right) \\ \geq 1 - 3 \exp \left(-n^{2\eta - 2\gamma + \theta_2} \right) - \exp \left(-\xi(1 + \xi)n^{\theta_2}(\log n)^2 \right) \end{aligned} \quad (1.20)$$

for all $n \geq N_3$.

Chromatic Number

We have the following result regarding the chromatic number for homogenous random graphs where each edge is independently open with probability r_n . We discuss separate cases depending on the asymptotic behaviour of r_n .

Theorem 3. (i) Suppose $r_n = \frac{1}{n^{\theta_2}}$ for some $0 < \theta_2 < \frac{1}{2}$. Fix $\xi, \zeta > 0$. There is a constant $N_1 = N_1(\xi, \theta_2) \geq 1$ so that

$$\begin{aligned} \mathbb{P} \left((1 - \xi) \frac{n^{1-\theta_2}}{2 \log n} \leq \chi(G(n, r_n)) \leq \frac{2(1 + \xi)}{1 - 2\theta_2} \frac{n^{1-\theta_2}}{\log n} \right) \\ \geq 1 - 3 \exp \left(-n^{1-\theta_2-\zeta} \right) - \exp \left(-\xi(1 + \xi)n^{\theta_2}(\log n)^2 \right) \end{aligned} \quad (1.21)$$

for all $n \geq N_1$.

(ii) Suppose $r_n = p$ for some $0 < p < 1$ and for all n . Fix $\xi, \zeta > 0$. There is a constant $N_2 = N_2(\xi, \zeta) \geq 1$ so that

$$\begin{aligned} \mathbb{P} \left((1 - \xi) \frac{n \log \left(\frac{1}{1-p} \right)}{2 \log n} \leq \chi(G(n, r_n)) \leq 2(1 + \xi) \frac{n \log \left(\frac{1}{1-p} \right)}{\log n} \right) \\ \geq 1 - 3 \exp \left(-n^{1-\zeta} \right) - \exp \left(-\frac{\xi(1 + \xi)}{\log \left(\frac{1}{1-p} \right)} (\log n)^2 \right) \end{aligned} \quad (1.22)$$

for all $n \geq N_2$.

(iii) Suppose $r_n = 1 - \frac{1}{n^{\theta_1}}$ for some $0 < \theta_1 < 1$. Fix $\xi, \zeta > 0$. There is a constant $N_3 = N_3(\xi, \zeta) \geq 1$ so that

$$\begin{aligned} \mathbb{P} \left((1 - \xi) \frac{\theta_1 n}{2 + \theta_1} \leq \chi(G(n, r_n)) \leq (1 + \xi) \frac{2\theta_1 n}{1 - \theta_1} \right) \\ \geq 1 - 3 \exp \left(-n^{2\eta - 2\gamma - \theta_1} \right) - n^{-\frac{\xi(1+\xi)}{\theta_1}} \end{aligned} \quad (1.23)$$

for all $n \geq N_3$.

The paper is organized as follows. In Section 2, we prove Proposition 1 and obtain preliminary estimates for proving the main Theorem 1. In Section 3, we prove Theorem 1 regarding the lower bound for clique numbers of inhomogenous graphs. In Section 4, we prove Proposition 2 and Theorem 2 for clique numbers of homogenous graphs. Finally in Section 5, we prove Theorem 3 regarding the chromatic number for homogenous graphs.

2 Preliminary estimates

For integer $q \geq 1$, let $G(q, \mathbf{p})$ be the random graph with vertex set $S_q = \{1, 2, \dots, q\}$. For integer $L \geq 2$, let $B_L(S_q)$ denote the event that the random graph $G(q, \mathbf{p})$ contains an open L -clique; i.e., there are vertices $\{v_i\}_{1 \leq i \leq L}$ such that the edge between v_i and v_j is open for any $1 \leq i \neq j \leq L$.

Proof of Proposition 1: We have

$$\mathbb{P}_{\mathbf{p}}(B_L(S_n)) \leq \sum_{S: \#S=L} \prod_{i,j \in S} p(i,j) = \sum_{S: \#S=L} \exp \left(- \sum_{i,j \in S} \log \left(\frac{1}{p(i,j)} \right) \right). \quad (2.1)$$

Setting $L = U_n$ and using the definition of t_n in (1.16), we have

$$\begin{aligned} \mathbb{P}_{\mathbf{p}}(B_L(S_q)) &\leq \sum_{S: \#S=U_n} \exp \left(- \binom{U_n}{2} \log \left(\frac{1}{t_n} \right) \right) \\ &\leq \binom{n}{U_n} \exp \left(- \binom{U_n}{2} \log \left(\frac{1}{t_n} \right) \right) \\ &\leq n^{U_n} \exp \left(- \binom{U_n}{2} \log \left(\frac{1}{t_n} \right) \right) \\ &= e^{-f_n U_n}, \end{aligned}$$

where f_n is as defined in (1.3). This proves the upper bound (1.17) in Proposition 1. ■

In what follows, we estimate the probability $\mathbb{P}(B_L^c(S_q))$ to obtain the lower bounds in Theorem 1. We use the following Binomial estimate. Let $\{X_i\}_{1 \leq i \leq m}$ be independent Bernoulli random variables with

$$\mathbb{P}(X_i = 1) = p_i = 1 - \mathbb{P}(X_i = 0).$$

We have the following Lemma.

Lemma 4. Fix $0 < \epsilon < \frac{1}{6}$. If

$$T_m = \sum_{i=1}^m X_i,$$

then

$$\mathbb{P}(|T_m - \mathbb{E}T_m| \geq \epsilon \mathbb{E}T_m) \leq \exp\left(-\frac{\epsilon^2 \mathbb{E}T_m}{4}\right) \quad (2.2)$$

for all $m \geq 1$.

For proof we refer to the Wikipedia link:

https://en.wikipedia.org/wiki/Chernoff_bound.

Small cliques estimate

For integer $q \geq 1$, we recall that $G(q, p)$ is the random graph with vertex set $S_q = \{1, 2, \dots, q\}$. For integer $L \geq 2$, let $B_L(S_q)$ denote the event that the random graph $G(q, \mathbf{p})$ contains an open L -clique; i.e., there are vertices $\{v_i\}_{1 \leq i \leq L}$ such that the edge between v_i and v_j is open for any $1 \leq i \neq j \leq L$. For $L \geq 2$, we define

$$t_L(q) = \sup_{\mathbf{p} \in \mathcal{N}(a, q, p_q)} \mathbb{P}_{\mathbf{p}}(B_L^c(S_q)). \quad (2.3)$$

We first obtain a recursive relation involving $t_L(q)$.

Lemma 5. Fix $0 < \epsilon < \frac{1}{6}$ and integer $L \geq 1$. For integer $q \geq 1$, suppose that $\mathbf{p} \in \mathcal{N}(a, q, p_q)$ (see (1.4)) and let

$$q_1 = [(p_q - \delta)(q - 1)] \quad (2.4)$$

be the largest integer less than or equal to $(p_q - \delta)(q - 1)$. Let $\delta \in \{p_q \epsilon, (1 - p_q) \epsilon\}$. For all integers q such that $q_1 \geq n^a$, we have that

$$t_L(q) \leq q t_{L-1}(q_1) + \exp\left(-\frac{\epsilon \delta}{10} q^2\right). \quad (2.5)$$

Proof of Lemma 5: For simplicity, we write $p = p_q$. We first prove that (2.5) is satisfied with $\delta = p \epsilon$. Let N_e be the number of open edges

in the random graph $G(q, \mathbf{p})$. Using (1.4), we have that $\mathbb{E}N_e \geq p\binom{q}{2}$. Fixing $0 < \epsilon < \frac{1}{6}$ and applying the binomial estimate (2.2) with $T_m = N_e$, we have that

$$\mathbb{P}_{\mathbf{p}} \left(N_e \geq p(1 - \epsilon) \binom{q}{2} \right) \geq 1 - \exp \left(-\frac{\epsilon^2 p}{4} \binom{q}{2} \right) = 1 - \exp \left(-\frac{\epsilon \delta}{4} \binom{q}{2} \right) \quad (2.6)$$

for all $q \geq 2$. The final term is obtained using $\delta = p\epsilon$. Using $\frac{1}{4}\binom{q}{2} \geq \frac{q^2}{10}$ for all $q \geq 5$ for the final term above we have

$$\mathbb{P}_{\mathbf{p}} \left(N_e \geq (p - \delta) \binom{q}{2} \right) \geq 1 - \exp \left(-\frac{\epsilon \delta}{10} q^2 \right). \quad (2.7)$$

Using (2.7), we therefore have

$$\mathbb{P}_{\mathbf{p}}(B_L^c(S_q)) = I_1 + I_2, \quad (2.8)$$

where

$$I_1 := \mathbb{P}_{\mathbf{p}} \left(B_L^c(S_q) \cap \left\{ N_e \geq (p - \delta) \binom{q}{2} \right\} \right) \quad (2.9)$$

and

$$\begin{aligned} I_2 &= \mathbb{P}_{\mathbf{p}} \left(B_L^c(S_q) \cap \left\{ N_e < (p - \delta) \binom{q}{2} \right\} \right) \\ &\leq \mathbb{P}_{\mathbf{p}} \left(N_e < (p - \delta) \binom{q}{2} \right) \\ &\leq \exp \left(-\frac{\epsilon \delta}{10} q^2 \right). \end{aligned} \quad (2.10)$$

We estimate I_1 as follows. Suppose that the event $N_e \geq (p - \delta)\binom{q}{2}$ occurs. If $d(v)$ denotes the degree of vertex $v \in \{1, 2, \dots, q\}$ in the random graph $G(q, \mathbf{p})$, we then have

$$\sum_{1 \leq v \leq q} d(v) = 2N_e \geq (p - \delta)q(q - 1).$$

In particular, there exists a vertex w such that

$$d(w) \geq (p - \delta)(q - 1) \geq q_1. \quad (2.11)$$

Here $q_1 = \lceil (p - \delta)(q - 1) \rceil$ is as defined in the statement of the Lemma. This implies that

$$\begin{aligned} & \mathbb{P}_{\mathbf{p}} \left(B_L^c(S_q) \cap \left\{ N_e \geq (p - \delta) \binom{q}{2} \right\} \right) \\ & \leq \mathbb{P}_{\mathbf{p}} \left(B_L^c(S_q) \cap \left(\bigcup_{1 \leq z \leq q} \{d(z) \geq q_1\} \right) \right) \\ & \leq \sum_{1 \leq z \leq q} \mathbb{P}_{\mathbf{p}} \left(B_L^c(S_q) \cap \{d(z) \geq q_1\} \right). \end{aligned} \quad (2.12)$$

Fixing $1 \leq z \leq q$, we evaluate each term in (2.12) separately. Letting $N(z) = N(z, G(q, \mathbf{p}))$ be the set of neighbours of z in the random graph $G(q, p)$, we have

$$\begin{aligned} & \mathbb{P}_{\mathbf{p}} \left(B_L^c(S_q) \cap \{d(z) \geq q_1\} \right) \\ & = \sum_{S: \#S \geq q_1, z \notin S} \mathbb{P}_{\mathbf{p}} \left(B_L^c(S_q) \cap \{N(z) = S\} \right). \end{aligned} \quad (2.13)$$

Suppose now that the event $B_L^c(S_q) \cap \{N(z) = S\}$ occurs for some fixed set S with $\#S \geq q_1$. We recall that since $B_L^c(S_q)$ occurs, there is no L -clique in the random graph $G(q, \mathbf{p})$ with vertex set $S_q = \{1, 2, \dots, q\}$. This means that there is no $(L - 1)$ -clique in the random induced subgraph of $G(q, \mathbf{p})$ formed by the vertices of S ; i.e., the event $B_{L-1}^c(S)$ occurs. Therefore we have

$$\begin{aligned} \mathbb{P}_{\mathbf{p}} \left(B_L^c(S_q) \cap \{N(z) = S\} \right) & \leq \mathbb{P}_{\mathbf{p}} \left(\{N(z) = S\} \cap B_{L-1}^c(S) \right) \\ & = \mathbb{P}_{\mathbf{p}}(N(z) = S) \mathbb{P}_{\mathbf{p}}(B_{L-1}^c(S)). \end{aligned} \quad (2.14)$$

The equality (2.14) true as follows. The event that $\{N(z) = S\}$ depends only on the state of edges containing z as an endvertex. On the other hand, the event $B_{L-1}^c(S)$ depends only on the state of edges having both their endvertices in S . Since the set S does not contain the vertex z (see (2.13)), we have that the events $\{N(z) = S\}$ and $B_{L-1}^c(S)$ are independent. This proves (2.14).

We obtain the desired recursion using (2.14) as follows. We recall that the set S contains at least q_1 vertices (see (2.13)). Therefore, setting T to be the set of the q_1 least indices in S , we have that if $B_{L-1}^c(S)$ occurs,

then $B_{L-1}^c(T)$ occurs; i.e., there is no $(L-1)$ -clique in the random induced subgraph formed by the vertices of T . From (2.14), we therefore have that

$$\begin{aligned}\mathbb{P}_{\mathbf{p}}\left(B_L^c(S_q) \cap \{N(z) = S\}\right) &\leq \mathbb{P}_{\mathbf{p}}(N(z) = S) \mathbb{P}_{\mathbf{p}}(B_{L-1}^c(T)) \\ &\leq \mathbb{P}_{\mathbf{p}}(N(z) = S) t_{L-1}(q_1).\end{aligned}\quad (2.15)$$

The final inequality is true as follows. Let \mathbf{p}_T be the vector formed by the probabilities $\{p(i, j)\}_{i, j \in T}$. From (1.4), we then have

$$\inf_{i \in T} \inf_S \frac{1}{\#S} \sum_{j \in S} p(i, j) \geq p \quad (2.16)$$

for all $n \geq N$. As in (1.4), the infimum is taken over all sets $S \subset T$ such that $\#S \geq n^a$ and $i \notin S$. This proves that $\mathbf{p}_T \in \mathcal{N}(a, q_1, p)$ and so (2.15) is true.

Substituting (2.15) into (2.13), we have

$$\begin{aligned}\mathbb{P}_{\mathbf{p}}\left(B_L^c(S_q) \cap \{d(z) \geq q_1\}\right) &\leq \sum_{S: \#S \geq q_1, z \notin S} \mathbb{P}_{\mathbf{p}}(N(z) = S) t_{L-1}(q_1) \\ &= \mathbb{P}_{\mathbf{p}}(\{d(z) \geq q_1\}) t_{L-1}(q_1) \\ &\leq t_{L-1}(q_1).\end{aligned}\quad (2.17)$$

$$\leq t_{L-1}(q_1). \quad (2.18)$$

The equality (2.17) is true since the events $\{N(z) = S\}$ are disjoint for distinct S . Substituting (2.18) into (2.12), we have

$$\mathbb{P}_{\mathbf{p}}\left(B_L^c(S_q) \cap \left\{N_e \geq (p - \delta) \binom{q}{2}\right\}\right) \leq \sum_{1 \leq z \leq q} t_{L-1}(q_1) = q t_{L-1}(q_1). \quad (2.19)$$

Using estimates (2.19) and (2.10) in (2.8) gives

$$\mathbb{P}_{\mathbf{p}}(B_L^c(S_q)) \leq q t_{L-1}(q_1) + \exp\left(-\frac{\epsilon \delta}{10} q^2\right), \quad (2.20)$$

for all q such that $q_1 \geq n^a$. Taking supremum over all $\mathbf{p} \in \mathcal{N}(a, q, p)$ proves (2.5) with $\delta = p\epsilon$.

It remains to see that (2.5) is satisfied with $\delta = \epsilon(1 - p)$. We recall that N_e denotes the number of open edges in the random graph $G(n, p)$ (see

the first paragraph of this proof). Let $W_e = \binom{n}{2} - N_e$ denote the number of closed edges. Fixing $0 < \epsilon < \frac{1}{6}$ and applying the binomial estimate (2.2) with $T_m = W_e$, we have that

$$\begin{aligned} \mathbb{P}_{\mathbf{p}} \left(W_e \leq (1-p)(1+\epsilon) \binom{q}{2} \right) &\geq 1 - \exp \left(-\frac{\epsilon^2(1-p)}{4} \binom{q}{2} \right) \\ &= 1 - \exp \left(-\frac{\epsilon\delta}{4} \binom{q}{2} \right) \end{aligned}$$

for all $q \geq 2$. The final estimate follows using $\delta = \epsilon(1-p)$. Since

$$\left\{ W_e \leq (1-p)(1+\epsilon) \binom{q}{2} \right\} = \left\{ N_e \geq (p-\delta) \binom{q}{2} \right\},$$

we again obtain (2.6). The rest of the proof is as above. \blacksquare

We use the recursion in the above Lemma iteratively to estimate the probability $t_L(q)$ of the event that there is no open L -clique in the random graph $G(q, \mathbf{p})$.

Lemma 6. *For integer $i \geq 1$, define*

$$v_i = v_i(q) = (p-\delta)^i q - \frac{1}{1-p+\delta}. \quad (2.21)$$

For all $q \geq 1$ such that $v_L(q) \geq n^a$, we have

$$t_L(q) \leq e^{-A_1} + 2e^{-A_2} \quad (2.22)$$

where

$$A_1 = -L \log q + \log \left(\frac{1}{1-p} \right) \frac{v_L^2}{4}. \quad (2.23)$$

and

$$A_2 = \frac{\epsilon\delta}{10} v_L^2 - L \log q. \quad (2.24)$$

To prove the above Lemma, we have a couple of preliminary estimates. Let $\{q_i\}_{0 \leq i \leq L}$ be integers defined recursively as follows. The term $q_0 = q$ and for $i \geq 1$, let

$$q_i = [(p-\delta)(q_{i-1} - 1)].$$

For a fixed $1 \leq i \leq L$, we have the following estimates.

(a1) We have

$$(p - \delta)(q_{i-1} - 1) - 1 \leq q_i \leq (p - \delta)q_{i-1} \leq q_{i-1} \leq q. \quad (2.25)$$

(a2) For $\delta > 0$, we have

$$v_i = (p - \delta)^i q - \frac{1}{1 - p + \delta} \leq q_i \leq (p - \delta)^i q. \quad (2.26)$$

Proof of (a1)–(a2): The property (a1) is obtained using the property $x - 1 \leq [x] \leq x$ for any $x > 0$. Applying the upper bound in (2.25) recursively, we get

$$q_i \leq (p - \delta)^i q_0 = (p - \delta)^i q.$$

This proves the upper bound in (2.26). For the lower bound we again proceed iteratively and obtain for $i \geq 2$ that

$$\begin{aligned} q_i &\geq (p - \delta)(q_{i-1} - 1) - 1 \\ &= (p - \delta)q_{i-1} - ((p - \delta) + 1) \\ &\geq (p - \delta)^2 q_{i-2} - ((p - \delta)^2 + (p - \delta) + 1) \\ &\quad \dots \\ &\geq (p - \delta)^i q_0 - \sum_{j=0}^i (p - \delta)^j. \end{aligned} \quad (2.27)$$

Since $\delta > 0$, we have

$$\sum_{j=0}^i (p - \delta)^j \leq \frac{1}{1 - p + \delta}$$

and so

$$q_i \geq (p - \delta)^i q_0 - \frac{1}{1 - p + \delta} = (p - \delta)^i q - \frac{1}{1 - p + \delta}.$$

This proves (a2). ■

Using the properties (a1)–(a2), we prove Lemma 6.

Proof of Lemma 6: Letting

$$r(q) = \exp\left(-\frac{\epsilon\delta}{10}q^2\right), \quad (2.28)$$

we apply the recursion (2.5) successively to get

$$\begin{aligned}
t_L(q) &\leq qt_{L-1}(q_1) + r(q) \\
&\leq q(q_1 t_{L-2}(q_2) + r(q_1)) + r(q) \\
&= qq_1 t_{L-2}(q_2) + qr(q_1) + r(q) \\
&\leq q^2 t_{L-2}(q_2) + qr(q_1) + r(q)
\end{aligned}$$

for all q such that $q_2 = q_2(q) \geq n^a$. The final estimate follows since $q_1 \leq q$ (see (2.25) of property (a1)). Proceeding iteratively, we obtain the following estimate for all q such that $q_{L-2}(q) \geq n^a$:

$$t_L(q) \leq J_1 + J_2, \quad (2.29)$$

where

$$J_1 := q^{L-2} t_2(q_{L-2}) \quad (2.30)$$

and

$$J_2 := \sum_{j=0}^{L-3} q^j r(q_j). \quad (2.31)$$

Let $v_L = v_L(q)$ be as defined in (2.21). For all q such that $v_L(q) \geq n^a$, we have the following bounds for the terms J_1 and J_2 .

$$J_1 \leq e^{-A_1} \quad (2.32)$$

and

$$J_2 \leq 2e^{-A_2}, \quad (2.33)$$

where A_1 and A_2 are as given in (2.23) and (2.24), respectively. This proves the Lemma.

Proof of (2.32) and (2.33): Since

$$q_j \geq q_L \geq v_L \quad (2.34)$$

for all $1 \leq j \leq L-1$ (property (a2)), the estimate (2.29) holds for all q such that $v_L = v_L(q) \geq n^a$.

We first evaluate J_1 . We have

$$J_1 \leq q^{L-2} t_2(q_{L-2}) \leq q^L t_2(q_{L-2}) \quad (2.35)$$

and

$$t_2(q_{L-2}) = (1-p)^{\binom{q_{L-2}}{2}} \leq (1-p)^{\binom{q_L}{2}} \leq (1-p)^{\binom{v_L}{2}}. \quad (2.36)$$

The first equality in (2.36) is true since there is no open 2-clique among a set of vertices if and only if all the edges between the vertices are closed. The second and third inequality follow from (2.34) and the fact that $1 - p < 1$. Substituting (2.36) into (2.35) we get the estimate (2.32) for the term J_1 .

For the second term J_2 , we argue as follows. The term $r(q) = \exp\left(-\frac{\epsilon\delta}{10}q^2\right)$ defined in (2.28) is decreasing in q . For $1 \leq j \leq L - 1$, we have from (2.34) that $q_j \geq v_L$ and so $r(q_j) \leq r(v_L)$. Using this in (2.31), we then have

$$J_2 \leq \left(\sum_{j=0}^{L-3} q^j\right) r(v_L) = \frac{q^{L-2} - 1}{q - 1} r(v_L) \leq 2q^{L-3} r(v_L) \leq 2q^L r(v_L). \quad (2.37)$$

The first inequality follows from the fact that $\frac{q^{L-2}-1}{q-1} \leq 2q^{L-3}$ for all $q \geq 3$. Using the expression for $r(q)$ in (2.37), we obtain (2.33). \blacksquare

3 Proof of Theorem 1

The following two estimates are used in what follows. For $0 < x < 1$, we have

$$-\log(1 - x) = \sum_{k \geq 1} \frac{x^k}{k} < \sum_{k \geq 1} x^k < \frac{x}{1 - x}, \quad (3.1)$$

and

$$-\log(1 - x) = \sum_{k \geq 1} \frac{x^k}{k} > x. \quad (3.2)$$

Proof of (i)

Here $\alpha_1 > 0$ and we use the estimates (2.23) and (2.24) of Lemma 6 to prove the Theorem 1. We first obtain a couple of additional estimates. Fix η and γ as in the statement of the Theorem. Also fix $\epsilon > 0$ small to be determined later and set $q_n = n$,

$$L_n = (1 - \eta) \frac{\log n}{\log\left(\frac{1}{p_n}\right)} \quad (3.3)$$

and

$$\delta_n = \epsilon p_n. \quad (3.4)$$

For a fixed $\epsilon > 0$, we have the following estimates regarding L_n and δ_n .

(b1) We have that

$$n^{-\alpha_1-\epsilon} \leq p_n \leq n^{-\alpha_1+\epsilon} \text{ and } \frac{1}{1-p_n} \leq \frac{1}{1-n^{-\alpha_1+\epsilon}} \leq 2 \quad (3.5)$$

and so

$$\alpha_2 = \limsup_n \frac{\log\left(\frac{1}{1-p_n}\right)}{\log n} = 0. \quad (3.6)$$

(b2) We have

$$L_n \leq \frac{1-\eta}{\alpha_1-\epsilon} \leq \frac{1}{\alpha_1-\epsilon} \quad (3.7)$$

for all n large.

(b3) There is a constant $N_0 = N_0(\eta, \epsilon) \geq 1$ such that

$$v_{L_n} \geq n^{\eta-2\epsilon} \quad (3.8)$$

for all n large.

Proof of (b1) – (b3): We prove (b1) first. We use the definition of $\alpha_1 > 0$ to get that

$$\frac{1}{\alpha_1 + \epsilon} \leq \frac{\log n}{\log\left(\frac{1}{p_n}\right)} \leq \frac{1}{\alpha_1 - \epsilon} \quad (3.9)$$

for all n large. This proves (3.5) and (3.6) in property (b1).

The inequality in (3.7) follows from the final estimate of (3.9) and the definition of L_n in (3.3). This proves (b2). To prove property (b3), we argue as follows. Setting $q = q_n = n$ and $L = L_n$ in the definition of v_i in (2.21), we then have

$$\begin{aligned} v_{L_n} &= \exp(L_n \log(p_n - \delta_n) + \log n) - \frac{1}{1 - p_n + \delta_n} \\ &\geq \exp(L_n \log(p_n - \delta_n) + \log n) - \frac{1}{1 - p_n} \\ &\geq e^{A_3} - 2 \end{aligned} \quad (3.10)$$

where

$$\begin{aligned} A_3 &= L_n \log(p_n - \delta_n) + \log n \\ &= L_n \log(p_n(1 - \epsilon)) + \log n \end{aligned} \quad (3.11)$$

$$\begin{aligned} &= L_n \log p_n + L_n \log(1 - \epsilon) + \log n \\ &= \eta \log n + L_n \log(1 - \epsilon) \end{aligned} \quad (3.12)$$

The final estimate in (3.10) follows from the final estimate in (3.5). The equality (3.11) above is obtained using $\delta_n = \epsilon p_n$ and the final equality (3.12) follows from the definition of L_n in (3.3).

Using $-\log(1-x) < \frac{x}{1-x}$ (see (3.1)) with $x = \epsilon$, we have

$$L_n \log(1-\epsilon) \geq \frac{-\epsilon}{1-\epsilon} L_n \geq \frac{-\epsilon}{1-\epsilon} \frac{1}{\alpha_1 - \epsilon}$$

where the final estimate follows using (3.7) in property (b2) above. Substituting the above into (3.12), we have

$$A_3 \geq \eta \log n - \frac{\epsilon}{1-\epsilon} \frac{1}{\alpha_1 - \epsilon} \geq (\eta - \epsilon) \log n \quad (3.13)$$

for all n large. Using (3.13) in (3.10), we have

$$v_{L_n} \geq n^{\eta-\epsilon} - 2 \geq n^{\eta-2\epsilon} \quad (3.14)$$

for all n large. ■

We use properties (b1) – (b3) to prove (i) in Theorem 1.

Proof of (i): From property (b3) and the choices of η and $\gamma > 0$ as in the statement of the Theorem, we have that $v_{L_n} \geq n^a$ for all $n \geq N_1$ large. Here $N_1 = N_1(\eta, \gamma, a)$ does not depend on the choice of \mathbf{p} . Thus the estimates for A_1 and A_2 in Lemma 6 are applicable.

Setting $q = q_n = n$ and $L = L_n$ and $\delta = \delta_n$ as in (3.3) and (3.4), respectively, in the expressions for A_1 and A_2 in (2.23) and (2.24), we have

$$A_1 = A_1(n) = -L_n \log n + \log \left(\frac{1}{1-p_n} \right) \frac{v_{L_n}^2}{4}, \quad (3.15)$$

and

$$A_2 = A_2(n) = \frac{\epsilon \delta_n}{10} v_{L_n}^2 - L_n \log n. \quad (3.16)$$

The following estimates for A_1 and A_2 imply the lower bound (1.8) for case (i) in Theorem 1. Fix $\gamma, \eta > 0$ as in the statement of the Theorem.

(c1) There are positive constants $\epsilon = \epsilon(\eta, \gamma) > 0$ and $M_1 = M_1(\eta, \gamma, \alpha_1) \geq 1$ so that

$$A_1 \geq n^{2\eta-2\gamma-\alpha_1} \quad (3.17)$$

for all $n \geq M_1$.

(c2) There are positive constants $\epsilon = \epsilon(\eta, \gamma) > 0$ and $M_2 = M_2(\eta, \gamma, \alpha_1) \geq 1$ so that

$$A_2 \geq n^{2\eta-2\gamma-\alpha_1} \quad (3.18)$$

for all $n \geq M_2$.

Proof of (c1) – (c2): We first prove (c1). Using the estimate (3.7) of property (b2), we have that the first term in (3.15) is

$$-L_n \log n \geq -\frac{\log n}{\alpha_1 - \epsilon} \quad (3.19)$$

and using estimate (3.8) of property (b3), we have that the second term is

$$\begin{aligned} \log \left(\frac{1}{1-p_n} \right) \frac{v_{L_n}^2}{4} &\geq \log \left(\frac{1}{1-p_n} \right) \frac{n^{2\eta-4\epsilon}}{4} \\ &\geq p_n \frac{n^{2\eta-4\epsilon}}{4} \end{aligned} \quad (3.20)$$

$$\geq \frac{n^{2\eta-5\epsilon-\alpha_1}}{4} \quad (3.21)$$

for all n large. The inequality (3.20) follows by setting $x = p_n$ in the estimate $-\log(1-x) > x$ (see (3.2)). The final estimate (3.21) follows from the first estimate (3.5) of property (b1).

Using estimates (3.21) and (3.19) in the expression for A_1 in (3.15), we have

$$\begin{aligned} A_1 &\geq \frac{1}{1-p_n} \left(\frac{n^{2\eta-5\epsilon-\alpha_1}}{4} - \frac{\log n}{\alpha_1 - \epsilon} \right) \\ &\geq \frac{1}{1-p_n} \left(\frac{n^{2\eta-5\epsilon-\alpha_1}}{5} \right) \\ &\geq \frac{1}{5} n^{2\eta-5\epsilon-\alpha_1} \end{aligned}$$

for all n large. The final estimate follows using $1-p_n < 1$. We now fix γ as in the statement of the Theorem and choose $\epsilon = \epsilon(\gamma, \eta) > 0$ small so that

$$\frac{1}{5} n^{2\eta-5\epsilon-\alpha_1} \geq n^{2\eta-2\gamma-\alpha_1}$$

for all n large. This proves (c1).

We prove (c2) as follows. Using the upper bound for $L_n \log n$ in (3.19) and the lower bound for v_{L_n} in property (b3), we have

$$\begin{aligned} A_2 &\geq \frac{\epsilon \delta_n}{10} \frac{n^{2\eta-4\epsilon}}{4} - \frac{\log n}{\alpha_1 - \epsilon} \\ &\geq \frac{\epsilon^2}{40} n^{2\eta-5\epsilon-\alpha_1} - \frac{\log n}{\alpha_1 - \epsilon} \end{aligned} \quad (3.22)$$

where the final estimate (3.22) follows from the fact that $\delta_n = \epsilon p_n$ and the lower bound for p_n in (3.5) (see property (b1)). As before, we choose $\epsilon = \epsilon(\eta, \gamma) > 0$ small so that the final term in (3.22) is at least $n^{2\eta-2\gamma-\alpha_1}$ for all n large. This proves (3.16). \blacksquare

Proof of (ii)

Fix η and γ as in the statement of the Theorem. Fix $\epsilon > 0$ small to be determined later and let $M = M(\epsilon) \geq 2$ be large so that

$$\frac{1 + \epsilon}{1 - \frac{1+\epsilon}{M}} < 1 + 2\epsilon. \quad (3.23)$$

Set $q_n = n$,

$$L_n = (1 - \eta) \frac{\log n}{\log \left(\frac{1}{p_n} \right)} \quad (3.24)$$

and

$$\delta_n = \left(\epsilon_1 p_n \mathbf{1} \left(p_n < 1 - \frac{1}{M} \right) + \epsilon (1 - p_n) \mathbf{1} \left(p_n \geq 1 - \frac{1}{M} \right) \right). \quad (3.25)$$

Here $\epsilon_1 = \epsilon_1(\epsilon) > 0$ is to be determined later. For a fixed $\epsilon > 0$, we have the following estimates regarding L_n and δ_n .

(b1) We have that

$$n^{-\epsilon} \leq p_n \leq 1 \text{ and } \frac{1}{1 - p_n} \leq n^\epsilon. \quad (3.26)$$

and

$$\delta_n \geq n^{-2\epsilon}. \quad (3.27)$$

for all n large.

(b2) We have

$$L_n \leq (1 - \eta) \frac{\log n}{1 - p_n} \leq \frac{\log n}{1 - p_n} \leq n^\epsilon \log n \quad (3.28)$$

for all n large. If $\epsilon_1 > 0$ is sufficiently small, then

$$R_n := \frac{\log \left(\frac{1}{p_n - \delta_n} \right)}{\log \left(\frac{1}{p_n} \right)} < 1 + 2\epsilon \quad (3.29)$$

for all n large.

(b3) There is a constant $N_0 = N_0(\eta, \epsilon) \geq 1$ such that

$$v_{L_n} \geq n^{\eta - 4\epsilon} \quad (3.30)$$

for all n large.

Proof of (b1) – (b3): The property (b1) is true as follows. Since $\alpha_1 = 0$, we have from (1.5) that $\log \left(\frac{1}{p_n} \right) \leq \epsilon \log n = \log(n^\epsilon)$ for all n large. This proves the first inequality in (3.26). Since $\alpha_2 = 0$, we have from (1.6) that $\log \left(\frac{1}{1 - p_n} \right) \leq \epsilon \log n = \log(n^\epsilon)$ for all n large. This proves the second inequality of (3.26).

To prove (3.27), we proceed as follows. If $\delta_n = \epsilon_1 p_n$, then we have from the first inequality in (3.26) that $\delta_n \geq \epsilon_1 n^{-\epsilon} \geq n^{-2\epsilon}$ for all n large. If $\delta_n = \epsilon(1 - p_n)$, then using the second inequality in (3.26), we have $\delta_n \geq \epsilon n^{-\epsilon} \geq n^{-2\epsilon}$ for all n large.

The first estimate (3.28) follows by using the lower bound $-\log(1 - x) > x$ with $x = 1 - p_n$ in the definition of L_n in (3.24). The second estimate in (3.28) follows using $\eta < 1$. The final estimate follows from (3.26). To prove (3.29), we consider two cases separately depending on whether $\delta_n = \epsilon p_n$ or $\delta_n = \epsilon(1 - p_n)$. If $\delta_n = \epsilon_1 p_n$, then $p_n < 1 - \frac{1}{M}$ and so we have

$$R_n = 1 + \frac{\log \left(\frac{1}{1 - \epsilon_1} \right)}{\log \left(\frac{1}{p_n} \right)} \leq 1 + \frac{\log \left(\frac{1}{1 - \epsilon_1} \right)}{\log \left(\frac{M}{M - 1} \right)} \leq 1 + \epsilon$$

if $\epsilon_1 = \epsilon_1(\epsilon) > 0$ is small.

If $\delta_n = \epsilon(1-p_n)$, then $p_n \geq 1 - \frac{1}{M}$ and $(1+\epsilon)(1-p_n) \leq \frac{1+\epsilon}{M} < 1$ since $M \geq 2$ and $0 < \epsilon < 1$. Therefore using the upper bound estimate $-\log(1-x) < \frac{x}{1-x}$ from (3.1) with $x = (1+\epsilon)(1-p_n)$, we have

$$-\log(p_n - \delta_n) = -\log(1 - (1+\epsilon)(1-p_n)) \leq \frac{(1+\epsilon)(1-p_n)}{1 - (1+\epsilon)(1-p_n)}.$$

Similarly using the lower bound estimate $-\log(1-x) > x$ from (3.2), we have

$$-\log p_n = -\log(1 - (1-p_n)) > 1 - p_n.$$

Using the above two estimates, we have

$$R_n \leq \frac{1+\epsilon}{1 - (1+\epsilon)(1-p_n)} \leq \frac{1+\epsilon}{1 - \frac{1+\epsilon}{M}} \leq 1 + 2\epsilon$$

by our choice of M from (3.23). This proves (b2).

To prove property (b3), we argue as follows. Setting $q_n = n, L = L_n$ (as in (3.24)) in the definition of v_i in (2.21) we have

$$\begin{aligned} v_{L_n} &= \exp(L_n \log(p_n - \delta_n) + \log n) - \frac{1}{1 - p_n + \delta_n} \\ &\geq \exp(L_n \log(p_n - \delta_n) + \log n) - \frac{1}{1 - p_n} \\ &= \frac{1}{1 - p_n} (e^{A_3} - 1) \end{aligned} \tag{3.31}$$

where

$$\begin{aligned} A_3 &= L_n \log(p_n - \delta_n) - \log\left(\frac{1}{1 - p_n}\right) + \log n \\ &\geq L_n \log(p_n - \delta_n) + (1 - \epsilon) \log n \end{aligned} \tag{3.32}$$

$$\geq (-(1 - \eta)(1 + 2\epsilon) + (1 - \epsilon)) \log n \tag{3.33}$$

$$= (\eta(1 + 2\epsilon) - 3\epsilon) \log n \tag{3.34}$$

$$\geq (\eta - 3\epsilon) \log n \tag{3.35}$$

for all n large. The estimate in (3.32) follows since α_2 defined in (1.5) is zero and so $\log\left(\frac{1}{1-p_n}\right) < \epsilon \log n$ for all n large. The estimate in (3.33) follows from (3.29) and the definition of L_n in (3.24).

Substituting (3.35) into (3.31), we get

$$v_{L_n} \geq \frac{1}{1-p_n} (n^{\eta-3\epsilon} - 1) \geq n^{\eta-3\epsilon} - 1 \geq n^{\eta-4\epsilon} \quad (3.36)$$

for all n large. The second inequality follows using $1-p_n < 1$. This proves (b3). ■

We use properties (b1) – (b3) to prove (ii) in Theorem 1.

Proof of (ii): We set $q = q_n = n$ and $L = L_n$ and $\delta = \delta_n$ as in (3.24) and (3.25), respectively, in the expressions for A_1 and A_2 in (2.23) and (2.24). We then have

$$A_1 = A_1(n) = -L_n \log n + \log \left(\frac{1}{1-p_n} \right) \frac{v_{L_n}^2}{4}, \quad (3.37)$$

and

$$A_2 = A_2(n) = \frac{\epsilon \delta_n}{10} v_{L_n}^2 - L_n \log n. \quad (3.38)$$

The following estimates for A_1 and A_2 imply the lower bound (1.10) for case (ii) in Theorem 1. Fix $\gamma, \eta > 0$ as in the statement of the Theorem.

(c1) There are positive constants $\epsilon = \epsilon(\gamma, \eta) > 0$ and $M_1 = M_1(\gamma, \eta, \alpha_1, \epsilon) \geq 1$ so that

$$A_1 \geq n^{2\eta-2\gamma} \quad (3.39)$$

for all $n \geq M_1$.

(c2) There are positive constants $\epsilon = \epsilon(\gamma, \eta) > 0$ and $M_2 = M_2(\gamma, \eta, \alpha_1, \epsilon) \geq 1$ so that

$$A_2 \geq n^{2\eta-2\gamma} \quad (3.40)$$

for all $n \geq M_2$.

Proof of (c1) – (c2): We first prove (c1). Using the estimate (3.28) of property (b2), we have that the first term in (3.37) is

$$-L_n \log n \geq -n^\epsilon (\log n)^2 \quad (3.41)$$

and using estimate (3.30) of property (b3), we have that the second term in (3.37) is

$$\begin{aligned} \log \left(\frac{1}{1-p_n} \right) \frac{v_{L_n}^2}{4} &\geq \log \left(\frac{1}{1-p_n} \right) \frac{n^{2\eta-8\epsilon}}{4} \\ &\geq p_n \frac{n^{2\eta-8\epsilon}}{4} \end{aligned} \quad (3.42)$$

$$\geq \frac{n^{2\eta-9\epsilon}}{4} \quad (3.43)$$

for all n large. The inequality (3.42) follows using $-\log(1-x) > x$ for $0 < x < 1$ (see (3.2)). The final estimate in (3.43) follows from the first estimate (3.26) of property (b1).

Using estimates (3.43) and (3.41) in (3.37), we have

$$\begin{aligned} A_1 &\geq \frac{n^{2\eta-9\epsilon}}{4} - n^\epsilon(\log n)^2 \\ &\geq n^{2\eta-2\gamma} \end{aligned} \quad (3.44)$$

for all n large provided $\epsilon = \epsilon(\eta, \gamma) > 0$ is small. This proves (c1).

We prove (c2) as follows. Using the upper bound for L_n in property (b2) and the lower bound for v_{L_n} in property (b3), we have

$$\begin{aligned} A_2 &\geq \frac{\epsilon \delta_n}{10} \frac{n^{2\eta-8\epsilon}}{4} - n^\epsilon(\log n)^2 \\ &= \frac{\epsilon}{40} n^{2\eta-10\epsilon} - n^\epsilon(\log n)^2 \\ &\geq n^{2\eta-2\gamma} \end{aligned} \quad (3.45)$$

for all n large, provided $\epsilon = \epsilon(\eta, \gamma) > 0$ is small. The estimate (3.45) follows from the estimate for δ_n in (3.27). This gives the estimate (c2) for the term A_2 . ■

Proof of (iii)

Fix η and γ as in the statement of the Theorem. Fix $\epsilon > 0$ small to be determined later and let $\alpha_2 = \alpha_2 > 0$ be as defined in (1.6). Set $q_n = n$,

$$L_n = (1 - \eta) \frac{\log n}{\log \left(\frac{1}{p_n} \right)} \quad (3.46)$$

and

$$\delta_n = \epsilon(1 - p_n). \quad (3.47)$$

For a fixed $\epsilon > 0$, we have the following estimates regarding L_n and δ_n .

(b1) We have that $\alpha_1 = 0$ and

$$n^{-\alpha_2-\epsilon} \leq 1 - p_n \leq n^{-\alpha_2+\epsilon} \text{ and } \frac{1}{p_n} \leq 2 \quad (3.48)$$

and

$$\delta_n \geq n^{-\alpha_2-2\epsilon} \quad (3.49)$$

for all n large.

(b2) We have

$$L_n \leq (1 - \eta - \alpha_2) \frac{\log n}{1 - p_n} \leq \frac{\log n}{1 - p_n} \leq n^{\alpha_2+\epsilon} \log n \quad (3.50)$$

and

$$R_n := \frac{\log\left(\frac{1}{p_n - \delta_n}\right)}{\log\left(\frac{1}{p_n}\right)} < 1 + 2\epsilon \quad (3.51)$$

for all n large.

(b3) We have that

$$v_{L_n} \geq n^{\eta+\alpha_2-5\epsilon} \quad (3.52)$$

for all n large.

Proof of (b1) – (b3): The property (b1) is true as follows. From the definition of $\alpha_2 > 0$ in (1.6) we have that

$$\alpha_2 - \epsilon \leq \frac{\log\left(\frac{1}{1-p_n}\right)}{\log n} \leq \alpha_2 + \epsilon \quad (3.53)$$

for all n large. This proves the first inequality in (3.48). The second inequality follows from the first inequality since

$$\frac{1}{p_n} \leq \frac{1}{1 - n^{-\alpha_2+\epsilon}} \leq 2$$

for all n large. This also proves that α_1 defined in (1.5) is zero. This proves the estimate (3.48) of property (b1). To prove (3.49), we use (3.48) and obtain

$$\delta_n = \epsilon(1 - p_n) \geq \epsilon n^{-\alpha_2-\epsilon} \geq n^{-\alpha_2-2\epsilon}$$

for all n large. This proves (b1).

The first estimate (3.50) follows by using the lower bound $-\log(1-x) > x$ with $x = 1-p_n$ in the definition of L_n in (3.24). The second estimate in (3.28) follows using $1 - \eta - \alpha_2 < 1$. The final estimate follows from (3.48). The

proof of (3.51) is analogous as the proof of (3.29) for the case $\delta_n = \epsilon(1 - p_n)$. This proves (b2).

To prove property (b3), we argue as follows. Setting $q_n = n, L = L_n$ (as in (3.46)) in the definition of v_i in (2.21) we have

$$\begin{aligned} v_{L_n} &= \exp(L_n \log(p_n - \delta_n) + \log n) - \frac{1}{1 - p_n + \delta_n} \\ &= \frac{1}{1 - p_n + \delta_n} (e^{A_3} - 1) \end{aligned} \quad (3.54)$$

where

$$\begin{aligned} A_3 &= L_n \log(p_n - \delta_n) - \log\left(\frac{1}{1 - p_n + \delta_n}\right) + \log n \\ &= L_n \log(p_n - \delta_n) + \log((1 + \epsilon)(1 - p_n)) + \log n. \end{aligned} \quad (3.55)$$

The final equality is true using $\delta_n = \epsilon(1 - p_n)$. For the middle term, we use the lower bound for $1 - p_n$ from (3.48) to get

$$\log((1 + \epsilon)(1 - p_n)) \geq \log(1 - p_n) \geq -(\alpha_2 + \epsilon) \log n. \quad (3.56)$$

We evaluate the first term in (3.55) as follows. Since $\alpha_2 < 1$, we have using (3.51) and the definition of L_n in (3.46) that

$$L_n \log(p_n - \delta_n) \geq -(1 - \eta)(1 + 2\epsilon) \log n. \quad (3.57)$$

Substituting (3.57) and (3.56) into (3.55), we have

$$\begin{aligned} A_3 &\geq -(1 - \eta - \alpha_2)(1 + 2\epsilon) \log n - (\alpha_2 + \epsilon) \log n + \log n \\ &= (\eta(1 + 2\epsilon) - 3\epsilon + 2\epsilon\alpha_2) \log n \\ &\geq (\eta - 3\epsilon) \log n \end{aligned} \quad (3.58)$$

for all n large. Substituting (3.58) into (3.54), we get

$$\begin{aligned} v_{L_n} &\geq \frac{1}{1 - p_n + \delta_n} (n^{\eta - 3\epsilon} - 1) \\ &\geq \frac{n^{\eta - 4\epsilon}}{1 - p_n + \delta_n} \\ &\geq \frac{n^{\eta - 4\epsilon}}{1 - p_n} \\ &\geq n^{\eta + \alpha_2 - 5\epsilon} \end{aligned} \quad (3.59)$$

for all n large. The final inequality follows from the estimate (3.48) in property (b1). This proves (b3) for the case $\alpha_2 < 1$. \blacksquare

We use properties (b1) – (b3) to prove (iii) in Theorem 1.

Proof of (iii): We set $q = q_n = n$ and $L = L_n$ and $\delta = \delta_n$ as in (3.46) and (3.47), respectively, in the expressions for A_1 and A_2 in (2.23) and (2.24). We then have

$$A_1 = A_1(n) = -L_n \log n + \log \left(\frac{1}{1 - p_n} \right) \frac{v_{L_n}^2}{4}, \quad (3.60)$$

and

$$A_2 = A_2(n) = \frac{\epsilon \delta_n}{10} v_{L_n}^2 - L_n \log n. \quad (3.61)$$

The following estimates for A_1 and A_2 imply the lower bound (1.12) in case (iii) of Theorem 1. Fix $\gamma, \eta > 0$ as in the statement of the Theorem.

(c1) There are positive constants $\epsilon = \epsilon(\gamma, \eta) > 0$ and $M_1 = M_1(\gamma, \eta, \alpha_1, \epsilon) \geq 1$ so that

$$A_1 \geq n^{2\eta - 2\gamma + 2\alpha_2} \quad (3.62)$$

for all $n \geq M_1$.

(c2) There are positive constants $\epsilon = \epsilon(\gamma, \eta) > 0$ and $M_2 = M_2(\gamma, \eta, \alpha_1, \epsilon) \geq 1$ so that

$$A_2 \geq n^{2\eta - 2\gamma + \alpha_2} \quad (3.63)$$

for all $n \geq M_2$.

Proof of (c1) – (c2): We first prove (c1). Using the estimate (3.50) of property (b2), we have that the first term in (3.60) is

$$-L_n \log n \geq -n^{\alpha_2 + \epsilon} (\log n)^2 \quad (3.64)$$

and using estimate (3.52) of property (b3), we have that the second term in (3.60) is

$$\begin{aligned} \log \left(\frac{1}{1 - p_n} \right) \frac{v_{L_n}^2}{4} &\geq \log \left(\frac{1}{1 - p_n} \right) \frac{n^{2\eta + 2\alpha_2 - 10\epsilon}}{4} \\ &\geq p_n \frac{n^{2\eta + 2\alpha_2 - 10\epsilon}}{4} \end{aligned} \quad (3.65)$$

$$\geq \frac{n^{2\eta + 2\alpha_2 - 10\epsilon}}{8} \quad (3.66)$$

for all n large. The inequality (3.65) follows using $-\log(1-x) > x$ for $0 < x < 1$ (see (3.2)). The final estimate in (3.66) follows from the final estimate (3.48) of property (b1).

Using estimates (3.66) and (3.64) in (3.60), we have

$$\begin{aligned} A_1 &\geq \frac{n^{2\eta+2\alpha_2-10\epsilon}}{8} - n^{\alpha_2+\epsilon}(\log n)^2 \\ &\geq n^{2\eta-2\gamma+2\alpha_2} \end{aligned} \quad (3.67)$$

for all n large, provided $\epsilon = \epsilon(\eta, \gamma) > 0$ is small. This proves (c1).

We prove (c2) as follows. Using the upper bound for L_n in property (b2) and the lower bound for v_{L_n} in property (b3), we have

$$\begin{aligned} A_2 &\geq \frac{\epsilon \delta_n n^{2\eta+2\alpha_2-10\epsilon}}{10 \cdot 4} - n^{\alpha_2+\epsilon}(\log n)^2 \\ &= \frac{\epsilon}{40} n^{2\eta+\alpha_2-12\epsilon} - n^{\alpha_2+\epsilon}(\log n)^2 \\ &\geq n^{2\eta-2\gamma+\alpha_2} \end{aligned} \quad (3.68)$$

for all n large, provided $\epsilon = \epsilon(\eta, \gamma) > 0$ is small. The estimate (3.68) follows from the estimate for δ_n in (3.49). This gives the estimate (c2) for the term A_2 . \blacksquare

4 Proof of Proposition 2 and Theorem 2

Proof of Proposition 2: By definition of α_2 in (1.6), we have

$$(\alpha_2 - \epsilon) \log n \leq \log \left(\frac{1}{1-p_n} \right) \leq (\alpha_2 + \epsilon) \log n$$

so that

$$\frac{1}{n^{\alpha_2+\epsilon}} \leq 1-p_n \leq \frac{1}{n^{\alpha_2-\epsilon}} \quad (4.1)$$

for all n large. Since $\alpha_2 > 2$, we fix $\epsilon > 0$ small so that $\alpha_2 - \epsilon > 2$. If N_e denote the number of open edges in the random graph $G(n, p_n)$, we then have

$$\mathbb{P}(N_e \geq 1) \leq \mathbb{E}N_e = p_n \binom{n}{2} \leq \frac{n^2}{n^{\alpha_2-\epsilon}} \longrightarrow 0 \quad (4.2)$$

as $n \rightarrow \infty$. But $\{N_e = 0\} = \{\omega(G(n, p_n)) = 1\}$ and so we obtain (1.13).

An analogous proof holds for the other case $\alpha_2 > 2$ by considering closed edges.

If $1 < \alpha_2 < 2$, we argue as follows. If W_e denotes the number of closed edges, then using the Binomial estimate (2.2), we have

$$\mathbb{P}(|W_e - \mathbb{E}W_e| \geq \epsilon \mathbb{E}W_e) \leq \exp\left(-\frac{\epsilon^2(1-p_n)}{4} \binom{n}{2}\right). \quad (4.3)$$

Using (4.1), we have

$$\mathbb{E}W_e = (1-p_n) \binom{n}{2} \leq \frac{1}{n^{\alpha_2-\epsilon}} \frac{n^2}{2} \leq \frac{1}{2} n^{2-\alpha_2+\epsilon} \quad (4.4)$$

and

$$\mathbb{E}W_e = (1-p_n) \binom{n}{2} \geq \frac{1}{n^{\alpha_2+\epsilon}} \frac{n^2}{4} \geq \frac{1}{4} n^{2-\alpha_2-\epsilon} \quad (4.5)$$

for all n large. The first inequality in (4.5) is obtained using (4.1) and $\binom{n}{2} \geq \frac{n^2}{4}$ for all n large. We choose $\epsilon > 0$ small so that

$$0 < 2 - \alpha_2 - 2\epsilon < 2 - \alpha_2 + 2\epsilon < 1.$$

We then have from (4.3), (4.5) and (4.4) that

$$\begin{aligned} \mathbb{P}\left(W_e \geq (1+\epsilon) \frac{1}{2} n^{2-\alpha_2+\epsilon}\right) &\leq \mathbb{P}(W_e \geq (1+\epsilon) \mathbb{E}W_e) \\ &\leq \exp\left(-\frac{\epsilon^2}{4} \frac{1}{4} n^{2-\alpha_2-\epsilon}\right) \\ &\leq \exp(-n^{2-\alpha_2-2\epsilon}) \end{aligned}$$

for all n large. Suppose now that the event $W_e \leq (1+\epsilon) \frac{1}{2} n^{2-\alpha_2+\epsilon}$ occurs and let \mathcal{S}_e be the set of all vertices belonging to the closed edges in the random graph $G(n, p_n)$. The induced subgraph G_S with vertex set $\{1, 2, \dots, n\} \setminus \mathcal{S}_e$ contains at least $n - (1+\epsilon) n^{2-\alpha_2+\epsilon}$ vertices and every edge in G_S is open. In other words, the graph G_S is an open clique containing at least

$$n - (1+\epsilon) n^{2-\alpha_2+\epsilon} \geq n - n^{2-\alpha_2+2\epsilon}$$

vertices, for all n large.

We now prove the upper bound (1.17). For integer $q \geq 1$, let $G(q, p)$ be the random graph with vertex set $S_q = \{1, 2, \dots, q\}$. For integer $L \geq 2$, let $B_L(S_q)$ denote the event that the random graph $G(q, p)$ contains an open L -clique; i.e., there are vertices $\{v_i\}_{1 \leq i \leq L}$ such that the edge between v_i and v_j is open for any $1 \leq i \neq j \leq L$. We have

$$\mathbb{P}(B_L(S_q)) \leq \binom{q}{L} p^{\binom{L}{2}} \leq q^L p^{\binom{L}{2}} = e^{-LA_0} \quad (4.6)$$

where

$$A_0 = A_0(q, p, L) = \left(\frac{L-1}{2} \right) \log \left(\frac{1}{p} \right) - \log q. \quad (4.7)$$

We now set $q = n, p = p_n$ and let $f_n \rightarrow \infty$ be any sequence as in the statement of the Theorem. Setting $L = U_n$ as defined in (1.16), we then have $A_0 = A_0(n) = f_n$. This proves the upper bound (1.17) in Lemma 2. ■

Proof of (i): Here α_1 defined in (1.5) equals θ_1 and α_2 as defined in (1.6) equals zero. The lower bound follows from (1.8), case (i) of Theorem 1. For the upper bound, we fix $\xi > 0$ and set $f_n = \xi \log n$ so that U_n as defined in (1.16) equals $\frac{2+\xi}{\theta_1} + 1$. The upper bound then follows from (1.17). ■

Proof of (ii): Here α_1 and α_2 defined in (1.5) and (1.6), respectively, both equal zero. Fixing η, γ as in the statement of (ii), the lower bound follows from (1.10), case (ii) of Theorem 1.

For the upper bound, we fix $0 < \xi_1 < \xi < 1$ and set $f_n = \xi_1 \log n$. The term U_n defined in (1.16) equals

$$U_n = \frac{(2 + \xi_1) \log n}{\log \left(\frac{1}{p} \right)} + 1 \leq \frac{(2 + \xi) \log n}{\log \left(\frac{1}{p} \right)}$$

for all $n \geq N_1$. Here $N_1 = N_1(\xi, \xi_1, p) \geq 1$ is a constant. Using (1.17) of Theorem 1, we have

$$\mathbb{P} \left(\omega(G(n, p)) \leq \frac{(2 + \xi) \log n}{\log \left(\frac{1}{p} \right)} \right) \geq 1 - \exp \left(- \frac{\xi_1 (2 + \xi_1)}{\log \left(\frac{1}{p} \right)} (\log n)^2 \right) \quad (4.8)$$

for all $n \geq N_1$. Choosing ξ_1 sufficiently close to ξ so that $\xi_1(2 + \xi_1) > \xi(1 + \xi)$, we obtain the upper bound in (ii). ■

Proof of (iii): Here α_1 defined in (1.5) equals zero and α_2 as defined in (1.6) equals θ_2 . Let $\eta, \gamma > 0$ be as in the statement of the Theorem and fix $\gamma_0, \eta_0 > 0$ such that $\gamma_0 - \frac{\theta_2}{2} < \gamma - \frac{\theta_2}{2} < \eta_0 < \eta < 1 - \theta_2$ and $\eta_0 - \gamma_0 > \eta - \gamma$. Let $\epsilon > 0$ be small to be determined later. Applying the lower bound (1.12), case (iii) in Theorem 1 with η_1 and γ_1 we have

$$\begin{aligned} \mathbb{P} \left(\omega(G(n, p_n)) \geq (1 - \theta_2 - \eta_0) \frac{\log n}{\log \left(\frac{1}{p_n} \right)} \right) \\ \geq 1 - 3 \exp \left(-n^{2\eta_0 - 2\gamma_0 + \theta_2} \right) \\ \geq 1 - 3 \exp \left(-n^{2\eta - 2\gamma + \theta_2} \right), \end{aligned} \quad (4.9)$$

where the final estimate follows from the choices of η_0 and γ_0 . We have

$$\log \left(\frac{1}{p_n} \right) = -\log(1 - (1 - p_n)) < \frac{1 - p_n}{1 - (1 - p_n)} \leq \frac{1 - p_n}{1 - \epsilon} = \frac{1}{n^{\theta_2}(1 - \epsilon)} \quad (4.10)$$

for all $n \geq N_1$. Here $N_1 = N_1(\epsilon) \geq 1$ is a constant. The first inequality in (4.10) follows from (3.1) and the second inequality follows from the fact that $1 - p_n < \epsilon$ for all $n \geq N_1$ large. From (4.10), we therefore have

$$\begin{aligned} (1 - \theta_2 - \eta_0) \frac{\log n}{\log \left(\frac{1}{p_n} \right)} &\geq (1 - \theta_2 - \eta_0)(1 - \epsilon)n^{\theta_2} \log n \\ &\geq (1 - \theta_2 - \eta)n^{\theta_2} \log n, \end{aligned} \quad (4.11)$$

provided $\epsilon = \epsilon(\eta_1, \eta, \theta_2) > 0$ is small. Fixing such an ϵ and substituting the estimate (4.11) into (4.9), we obtain the lower bound in (1.20).

For the upper bound, we fix $\xi > 0$ and set $f_n = \xi \log n$. The term U_n defined in (1.16) is then

$$U_n = \frac{(2 + \xi) \log n}{\log \left(\frac{1}{p_n} \right)} \quad (4.12)$$

and using (3.2), we have

$$\log \left(\frac{1}{p_n} \right) = -\log(1 - (1 - p_n)) > 1 - p_n = \frac{1}{n^{\theta_2}}. \quad (4.13)$$

Using the bounds (4.10) and (4.13) in (4.12), we have

$$(2 + \xi)(1 - \epsilon)n^{\theta_2} \log n \leq U_n \leq (2 + \xi)n^{\theta_2} \log n \quad (4.14)$$

for all $n \geq N_1$. Here $N_1 = N_1(\epsilon) \geq 1$ is the constant in (4.10).

Using the above bounds in the upper bound (1.17) of Theorem 1, we have

$$\mathbb{P}(\omega(G(n, p_n)) \leq (2 + \xi)n^{\theta_2} \log n) \geq 1 - \exp(-\xi(2 + \xi)(1 - \epsilon)n^{\theta_2}(\log n)^2) \quad (4.15)$$

for all $n \geq N_1$. Choosing $\epsilon > 0$ small so that $(2 + \xi)(1 - \epsilon) > 1 + \xi$, we obtain the upper bound in (1.20). \blacksquare

5 Proof of Theorem 3

For a graph $G = (V, E)$ on n vertices, let $\alpha(G)$ be the independence number of the graph G defined as follows. For integer $0 \leq h \leq n$, we say that $\alpha(G) = h$ if and only if the following two conditions are satisfied:

- (a) There is a set of h vertices, none of which have an edge between them.
- (b) If $h + 1 \leq n$, then every set of $h + 1$ vertices have an edge between them.

As in Section 1, let $\omega(G)$ denote the clique number of G . Let $\overline{G} = (\overline{V}, \overline{E})$ denote the complement of the graph G defined as follows. The vertex set $\overline{V} = V$ and an edge $e \in \overline{E}$ if and only if $e \notin E$. The following three properties are used to prove Theorem 3.

(d1) We have

$$\alpha(G) = \omega(\overline{G}). \quad (5.1)$$

(d2) We have

$$\chi(G) \geq \frac{n}{\alpha(G)} = \frac{n}{\omega(\overline{G})}. \quad (5.2)$$

(d3) Suppose for some integer $1 \leq m \leq n$, every set of m vertices in the complement graph \overline{G} contains a clique of size L . We then have

$$\chi(G) \leq \frac{n - m}{L} + m + 1 \leq \frac{n}{L} + 2m. \quad (5.3)$$

The lower bounds in Theorem 3, follow from the respective upper bounds (1.18), (1.19) and (1.20) on the clique number $\omega(G(n, 1 - r_n))$ of Theorem 2 and property (d2) above. This is because, the random graph $\overline{G}(n, r_n)$ has the same distribution as the random graph $G(n, 1 - r_n)$.

For the upper bounds, we consider each case separately.

Proof of (i): Here $r_n = \frac{1}{n^{\theta_2}}$ for some $\theta_2 > 0$. To estimate the chromatic

number using property (d3), we identify cliques in subsets of the random graph $G(n, 1 - r_n)$. Fix $\beta > 0$ to be determined later and set $m = n^{1-\beta}$ and apply Theorem 2, case (iii) for the random graph $G(m, p_m)$, where

$$p_m = 1 - \frac{1}{m^{\theta_{22}}} = 1 - r_n, \quad (5.4)$$

where $\theta_{22} = \frac{\theta_2}{1-\beta}$. We then have $\alpha_1 = 0$ and $\alpha_2 = \theta_{22}$, where α_1 and α_2 are as defined in (1.5) and (1.6), respectively. Let $\eta, \gamma > 0$ be such that

$$\frac{1 - \theta_{22}}{2} + \gamma < \eta < 1 - \theta_{22} \quad (5.5)$$

From the proof of lower bound of (1.20), there is a positive integer $N_3 = N_3(\eta, \gamma, \xi) \geq 1$ so that

$$\mathbb{P}(\omega(G(m, p_m)) \leq L) \leq 3 \exp(-m^{2\eta-2\gamma+\theta_{22}}) \quad (5.6)$$

for all m large, where

$$L = (1 - \eta - \theta_{22})m^{\theta_{22}} \log m = (1 - \eta - \theta_{22})(1 - \beta)n^{\theta_2} \log n. \quad (5.7)$$

The final estimate above follows using $m = n^{1-\beta}$.

Let \mathcal{S}_m be the set of subsets of size m in $\{1, 2, \dots, n\}$ and for a set $S \in \mathcal{S}_m$, let $F_n(S)$ denote the event that the random induced subgraph of $G(n, 1 - r_n)$ with vertex set S contains an open L -clique. From (5.6) we have that

$$\mathbb{P}(F_n^c(S)) \leq 3 \exp(-m^{2\eta-2\gamma-\theta_{22}})$$

for all n large. Let $F_n = \bigcap_{S \in \mathcal{S}_m} F_n(S)$ denote the event that every set of m vertices in the random graph $G(n, 1 - r_n)$ contains an L -clique. Since there are $\binom{n}{m}$ sets in \mathcal{S}_m , we have

$$\mathbb{P}(F_n^c) \leq \binom{n}{m} 3 \exp(-m^{2\eta-2\gamma+\theta_{22}}) \leq n^m 3 \exp(-m^{2\eta-2\gamma+\theta_{22}}) = 3e^{-B}, \quad (5.8)$$

where

$$B = m^{2\eta-2\gamma-\theta_{22}} - m \log n = m^{2\eta-2\gamma-\theta_{22}} - \frac{m}{1-\beta} \log m. \quad (5.9)$$

The final estimate follows since $m = n^{1-\beta}$. From the choices of η and γ in (5.5), we have that $2\eta - 2\gamma + \theta_{22} > 1$ and so

$$B \geq \frac{1}{2} m^{2\eta-2\gamma+\theta_{22}} = \frac{1}{2} n^{(1-\beta)(2\eta-2\gamma)+\theta_2} \quad (5.10)$$

for all $n \geq N_2$. Here $N_2 = N_2(\eta, \gamma, \beta) \geq 1$ is a constant and the final equality follows from the definition of $m = n^{1-\beta}$ and $\theta_{22} = \frac{\theta_2}{1-\beta}$ above.

If the event F_n occurs, then using property (d3), we have that

$$\chi(G(n, r_n)) \leq \frac{n}{L} + 2m = \frac{1}{(1-\eta-\theta_{22})(1-\beta)} \frac{n^{1-\theta_2}}{\log n} + 2n^{1-\beta} \quad (5.11)$$

Fixing $\beta > \theta_2$ and $\xi > 0$, we have that the final expression in (5.11) is at most

$$\left(\frac{1 + 0.5\xi}{(1-\eta-\theta_{22})(1-\beta)} \right) \frac{n^{1-\theta_2}}{\log n}$$

for all n large. Summarizing, we have from (5.9), (5.10) and (5.8) that

$$\begin{aligned} \mathbb{P} \left(\chi(G(n, r_n)) \leq \left(\frac{1 + 0.5\xi}{(1-\eta-\theta_{22})(1-\beta)} \right) \frac{n^{1-\theta_2}}{\log n} \right) \\ \geq 1 - 3 \exp \left(-\frac{1}{2} n^{(1-\beta)(2\eta-2\gamma)+\theta_2} \right). \end{aligned} \quad (5.12)$$

We have the following property.

(f1) Let

$$\mathcal{T} = \{(\eta, \gamma, \beta) : \beta > \theta_2 \text{ and (5.5) is satisfied}\}.$$

Let $\xi, \zeta > 0$ be as in the statement of the Theorem. There exists $(\eta, \gamma, \beta) \in \mathcal{T}$ such that

$$\frac{1 + 0.5\xi}{(1-\eta-\theta_{22})(1-\beta)} \leq \frac{2(1+\xi)}{1-2\theta_2} \quad (5.13)$$

and

$$(1-\beta)(2\eta-2\gamma)+\theta_2 \geq 1-\theta_2-\zeta. \quad (5.14)$$

This proves the upper bound in (1.21) in Theorem 3.

Proof of (f1): We recall that $\theta_{22} = \frac{\theta_2}{1-\beta}$ and we have the constraint that $\beta > \theta_2$ and $\gamma > 0$. Since

$$\inf_{\beta > \theta_2, \gamma > 0} \left(\gamma + \frac{1-\theta_{22}}{2} \right) = \frac{1}{2} \left(1 - \frac{\theta_2}{1-\theta_2} \right) = \frac{1-2\theta_2}{2(1-\theta_2)} \quad (5.15)$$

we have from (5.5) that the least possible value for η is $\frac{1-2\theta_2}{2(1-\theta_2)}$; i.e., if $(\eta, \gamma, \beta) \in \mathcal{T}$, then $\eta \geq \frac{1-2\theta_2}{2(1-\theta_2)}$ and $\beta > \theta_2$.

Fix $\delta > 0$ small and fix $\theta_2 < \beta < \theta_2 + \delta$ and $\frac{1-2\theta_2}{2(1-\theta_2)} < \eta < \frac{1-2\theta_2}{2(1-\theta_2)} + \delta$. From (5.15), we have that if $\delta, \gamma > 0$ are sufficiently small, then $(\eta, \gamma, \beta) \in \mathcal{T}$. Also, we have that $(1 - \eta - \theta_{22})(1 - \beta) \geq f(\theta_2, \delta)$, where

$$f(\theta_2, \delta) = \left(1 - \frac{1 - 2\theta_2}{2(1 - \theta_2)} - \delta - \frac{\theta_2}{1 - \theta_2 - \delta}\right) (1 - \theta_2 - \delta).$$

Since $f(\theta_2, 0) = \frac{1-2\theta_2}{2}$, we fix $\xi > 0$ and choose $\delta = \delta(\xi) > 0$ smaller if necessary so that

$$(1 - \theta_{22} - \eta)(1 - \beta) \geq \frac{1 - 2\theta_2}{2} \frac{1 + 0.5\xi}{1 + \xi}.$$

This proves (5.13).

For (5.14), we proceed analogously and use the fact that $\eta \geq \frac{1-2\theta_2}{2(1-\theta_2)}$ to obtain

$$(1 - \beta)(2\eta - 2\gamma) + \theta_2 \geq (1 - \theta_2 - \delta) \left(\frac{1 - 2\theta_2}{1 - \theta_2} - 2\gamma \right) + \theta_2 \geq 1 - \theta_2 - \zeta,$$

provided $\delta, \gamma > 0$ are small. ■

Proof of (ii): Here $r_n = p$ for some $p \in (0, 1)$ and for all $n \geq 2$. As in the proof of (i) above, we identify cliques in subsets of the random graph $G(n, 1 - r_n)$. Fix $\beta > 0$ to be determined later and set $m = n^{1-\beta}$, $p_m = 1 - p$ and apply Theorem 2, case (ii) for the random graph $G(m, 1 - p)$. We then have $\alpha_1 = \alpha_2 = 0$, where α_1 and α_2 are as defined in (1.5) and (1.6), respectively. Fixing

$$\frac{1 + \gamma}{2} < \eta < 1, \tag{5.16}$$

we have from the proof of lower bound of (1.19), that there is a positive integer $N = N(\eta, \gamma) \geq 1$ so that

$$\mathbb{P}(\omega(G(m, 1 - p)) \leq L) \leq 3 \exp(-m^{2\eta-2\gamma}) \tag{5.17}$$

for all $m \geq N_3$, where

$$L = (1 - \eta) \frac{\log m}{\log\left(\frac{1}{1-p}\right)} = (1 - \eta)(1 - \beta) \frac{\log n}{\log\left(\frac{1}{1-p}\right)}. \tag{5.18}$$

The final estimate above follows using $m = n^{1-\beta}$.

As in case (i), let F_n denote the event that every set of m vertices in the random graph $G(n, 1 - r_n)$ contains an L -clique. Analogous to (5.8), we then have

$$\mathbb{P}(F_n^c) \leq \binom{n}{m} 3 \exp(-m^{2\eta-2\gamma}) \leq n^m 3 \exp(-m^{2\eta-2\gamma}) = 3e^{-B}, \quad (5.19)$$

where

$$B = m^{2\eta-2\gamma} - m \log n = m^{2\eta-2\gamma} - \frac{m}{1-\beta} \log m. \quad (5.20)$$

The final estimate follows since $m = n^{1-\beta}$. From the choices of η and γ in (5.16), we have that $2\eta - 2\gamma > 1$ and so

$$B \geq \frac{1}{2} m^{2\eta-2\gamma} = \frac{1}{2} n^{(1-\beta)(2\eta-2\gamma)} \quad (5.21)$$

for all $n \geq N_2$. Here $N_2 = N_2(\eta, \gamma, \beta) \geq 1$ is a constant and the final equality follows from the definition of $m = n^{1-\beta}$.

If the event F_n occurs, then using property (d3), we have that

$$\chi(G(n, r_n)) \leq \frac{n-m}{L} + m \leq \frac{n}{L} + m = \frac{\log\left(\frac{1}{1-p}\right)}{(1-\eta)(1-\beta) \log n} n + n^{1-\beta} \quad (5.22)$$

Fixing $\beta > 0$ and $\xi > 0$, we have that the final term in (5.11) is at most

$$\left(\frac{1 + 0.5\xi}{(1-\eta)(1-\beta)} \right) \frac{n \log\left(\frac{1}{1-p}\right)}{\log n}$$

for all $n \geq N_3$. Here $N_3 = N_3(\eta, \beta, \xi) \geq 1$ is a constant. Summarizing, we have from (5.20), (5.21) and (5.19) that

$$\begin{aligned} \mathbb{P} \left(\chi(G(n, r_n)) \leq \left(\frac{1 + 0.5\xi}{(1-\eta)(1-\beta)} \right) \frac{n \log\left(\frac{1}{1-p}\right)}{\log n} \right) \\ \geq 1 - 3 \exp \left(-\frac{1}{2} n^{(1-\beta)(2\eta-2\gamma)} \right) \end{aligned} \quad (5.23)$$

Analogous to property (f1) above, we have the following property.

(f2) Let

$$\mathcal{T} = \{(\eta, \gamma, \beta) : \beta > 0 \text{ and (5.16) is satisfied}\}$$

and fix $\zeta > 0$. There exists $(\eta, \gamma, \beta) \in \mathcal{T}$ so that

$$\frac{1 + 0.5\xi}{(1 - \eta)(1 - \beta)} \leq 2(1 + \xi) \quad (5.24)$$

and

$$(1 - \beta)(2\eta - 2\gamma) \geq 1 - \zeta. \quad (5.25)$$

Substituting the above into (5.23), we obtain the upper bound in (1.22) in Theorem 3.

Proof of (f2): From (5.16), we have that the minimum possible value for η is $\frac{1}{2}$. Choosing $\eta > \frac{1}{2}$ sufficiently close to $\frac{1}{2}$ and $\beta > 0$ small, both (5.24) and (5.25) are satisfied. ■

Proof of (iii): Here $r_n = 1 - \frac{1}{n^{\theta_1}}$ for some $\theta_1 > 0$. As before, we identify cliques in subsets of the random graph $G(n, 1 - r_n)$. Fix $\beta > 0$ to be determined later and set $m = \beta n$ and apply Theorem 2, case (iii) for the random graph $G(m, p_m)$, where

$$p_m = 1 - \frac{\beta^{\theta_1}}{m^{\theta_1}} = 1 - r_n. \quad (5.26)$$

We then have $\alpha_1 = \theta_1$ and $\alpha_2 = 0$, where α_1 and α_2 are as defined in (1.5) and (1.6), respectively. Let $\eta, \gamma > 0$ be such that

$$\frac{1 + \theta_1}{2} + \gamma < \eta < 1. \quad (5.27)$$

From the proof of lower bound of (1.18), we have that

$$\mathbb{P}(\omega(G(m, p_m)) \leq L) \leq 3 \exp(-m^{2\eta - 2\gamma - \theta_1}) \quad (5.28)$$

for all m large, where

$$L = \frac{2(1 - \eta)}{\theta_1}. \quad (5.29)$$

As in cases (i) – (ii), let F_n denote the event that every set of m vertices in the random graph $G(n, 1 - r_n)$ contains an L -clique. Using $\binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$ for integers $1 \leq k \leq n$, we have

$$\mathbb{P}(F_n^c) \leq \binom{n}{m} 3 \exp(-m^{2\eta - 2\gamma - \theta_1}) \leq \left(\frac{ne}{m}\right)^m 3 \exp(-m^{2\eta - 2\gamma - \theta_1}) = 3e^{-B}, \quad (5.30)$$

where

$$B = m^{2\eta-2\gamma-\theta_1} - m \log \left(\frac{e}{\beta} \right). \quad (5.31)$$

The final estimate in (5.30) follows since $m = \beta n$. From the choices of η and γ in (5.27), we have that $2\eta - 2\gamma - \theta_1 > 1$ and so

$$B \geq \frac{1}{2} m^{2\eta-2\gamma-\theta_1} = \frac{1}{2} (\beta n)^{2\eta-2\gamma-\theta_1} \quad (5.32)$$

for all $n \geq N_2$. Here $N_2 = N_2(\eta, \gamma, \beta) \geq 1$ is a constant and the final equality follows from the definition of $m = \beta n$.

If the event F_n occurs, then using property (d3), we have that

$$\chi(G(n, r_n)) \leq \frac{n}{L} + 2m \leq \frac{\theta_1}{2(1-\eta)} n + \beta n + 1. \quad (5.33)$$

Summarizing, we have from (5.32) and (5.30) that

$$\mathbb{P} \left(\chi(G(n, r_n)) \leq \frac{\theta_1}{2(1-\eta)} n + \beta n + 1 \right) \geq 1 - \exp \left(-\frac{1}{2} (\beta n)^{2\eta-2\gamma-\theta_1} \right) \quad (5.34)$$

for all n large. We have the following property.

(f3) Fix $\xi, \zeta > 0$ and let

$$\mathcal{T} = \{(\eta, \gamma, \beta) : \text{The condition (5.27) is satisfied}\}.$$

There exists $(\eta, \gamma, \beta) \in \mathcal{T}$ such that

$$\frac{\theta_1}{2(1-\eta)} + \beta \leq \frac{\theta_1}{1-\theta_1} (1 + \xi) \quad (5.35)$$

and

$$\frac{1}{2} (\beta n)^{2\eta-2\gamma-\theta_1} \geq n^{1-\theta_1-\zeta} \quad (5.36)$$

for all n large. Using the above estimates in (5.33), we obtain the upper bound in (1.23) in Theorem 3.

Proof of (f3): From (5.27), we have that the least possible value for η is $\frac{1+\theta_1}{2}$. Fix $\zeta, \xi > 0$. Choosing $\gamma, \beta > 0$ sufficiently small and $\eta > \frac{1+\theta_1}{2}$ sufficiently close of $\frac{1+\theta_1}{2}$, we get that

$$\frac{\theta_1}{2(1-\eta)} + \beta \leq \frac{\theta_1}{1-\theta_1} (1 + \xi)$$

and $2\eta - 2\gamma - \theta_1 > 1 - \zeta$. This proves (5.35) and (5.36). ■

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